To estimate the accuracy, we note that $|x_2 - x_1| < 0.2$. Thus, after n steps it follows from Corollary 3.5.10(i) that we are sure that $|x^* - x_n| \le 3^{n-1}(7^{n-2} \cdot 20)$. Thus, when $n = 6$, we are sure that

$$|x^* - x_6| \le 3^5/(7^4 \cdot 20) = 243/48\,020 < 0.0051.$$

Actually the approximation is substantially better than this. In fact, since $|x_6 - x_5| < 0.000\,0005$, it follows from 3.5.10(ii) that $|x^* - x_6| \le \frac{3}{4}|x_6 - x_5| < 0.000\,0004$. Hence the first five decimal places of x_6 are correct. □

Exercises for Section 3.5

1. Give an example of a bounded sequence that is not a Cauchy sequence.
2. Show directly from the definition that the following are Cauchy sequences.
   (a) $\left(\dfrac{n+1}{n}\right)$, (b) $\left(1 + \dfrac{1}{2!} + \cdots + \dfrac{1}{n!}\right)$.
3. Show directly from the definition that the following are not Cauchy sequences.
   (a) $((-1)^n)$, (b) $\left(n + \dfrac{(-1)^n}{n}\right)$, (c) $(\ln n)$
4. Show directly from the definition that if (x_n) and (y_n) are Cauchy sequences, then $(x_n + y_n)$ and $(x_n y_n)$ are Cauchy sequences.
5. If $x_n := \sqrt{n}$, show that (x_n) satisfies $\lim|x_{n+1} - x_n| = 0$, but that it is not a Cauchy sequence.
6. Let p be a given natural number. Give an example of a sequence (x_n) that is not a Cauchy sequence, but that satisfies $\lim|x_{n+p} - x_n| = 0$.
7. Let (x_n) be a Cauchy sequence such that x_n is an integer for every $n \in \mathbb{N}$. Show that (x_n) is ultimately constant.
8. Show directly that a bounded, monotone increasing sequence is a Cauchy sequence.
9. If $0 < r < 1$ and $|x_{n+1} - x_n| < r^n$ for all $n \in \mathbb{N}$, show that (x_n) is a Cauchy sequence.
10. If $x_1 < x_2$ are arbitrary real numbers and $x_n := \frac{1}{2}(x_{n-2} + x_{n-1})$ for $n > 2$, show that (x_n) is convergent. What is its limit?
11. If $y_1 < y_2$ are arbitrary real numbers and $y_n := \frac{1}{3}y_{n-1} + \frac{2}{3}y_{n-2}$ for $n > 2$, show that (y_n) is convergent. What is its limit?
12. If $x_1 > 0$ and $x_{n+1} := (2 + x_n)^{-1}$ for $n \ge 1$, show that (x_n) is a contractive sequence. Find the limit.
13. If $x_1 := 2$ and $x_{n+1} := 2 + 1/x_n$ for $n \ge 1$, show that (x_n) is a contractive sequence. What is its limit?
14. The polynomial equation $x^3 - 5x + 1 = 0$ has a root r with $0 < r < 1$. Use an appropriate contractive sequence to calculate r within 10^{-4}.

Section 3.6 Properly Divergent Sequences

For certain purposes it is convenient to define what is meant for a sequence (x_n) of real numbers to "tend to $\pm\infty$."